

On the properties of even and odd sequences

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Abstract

In this paper we continue to investigate the properties of those sequences $\{a_n\}$ satisfying the condition $\sum_{k=0}^n \binom{n}{k} (-1)^k a_k = \pm a_n$ ($n \geq 0$). As applications we deduce new recurrence relations and congruences for Bernoulli and Euler numbers.

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1. Introduction

The classical binomial inversion formula states that $a_n = \sum_{k=0}^n \binom{n}{k} (-1)^k b_k$ ($n = 0, 1, 2, \dots$) if and only if $b_n = \sum_{k=0}^n \binom{n}{k} (-1)^k a_k$ ($n = 0, 1, 2, \dots$). Following [8] we continue to study those sequences $\{a_n\}$ with the property $\sum_{k=0}^n \binom{n}{k} (-1)^k a_k = \pm a_n$ ($n = 0, 1, 2, \dots$).

Definition 1.1. *If a sequence $\{a_n\}$ satisfies the relation*

$$\sum_{k=0}^n \binom{n}{k} (-1)^k a_k = a_n \quad (n = 0, 1, 2, \dots),$$

we say that $\{a_n\}$ is an even sequence. If $\{a_n\}$ satisfies the relation

$$\sum_{k=0}^n \binom{n}{k} (-1)^k a_k = -a_n \quad (n = 0, 1, 2, \dots),$$

we say that $\{a_n\}$ is an odd sequence.

From [8, Theorem 3.2] we know that $\{a_n\}$ is an even (odd) sequence if and only if $e^{-x/2} \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$ is an even (odd) function. Throughout this paper, S^+ denotes the set of even sequences, and S^- denotes the set of odd sequences. In [8] the author stated that

$$\left\{ \frac{1}{2^n} \right\}, \left\{ \binom{n+2m-1}{m}^{-1} \right\}, \left\{ \binom{2n}{n} 2^{-2n} \right\}, \left\{ (-1)^n \int_0^{-1} \binom{x}{n} dx \right\} \in S^+.$$

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Let $\{B_n\}$ be the Bernoulli numbers given by $B_0 = 1$ and $\sum_{k=0}^{n-1} \binom{n}{k} B_k = 0$ ($n \geq 2$). It is well known that $B_1 = -\frac{1}{2}$ and $B_{2m+1} = 0$ for $m \geq 1$. Thus,

$$\sum_{k=0}^n \binom{n}{k} (-1)^k \cdot (-1)^k B_k = B_n + \sum_{k=0}^{n-1} \binom{n}{k} B_k = (-1)^n B_n$$

and so $\{(-1)^n B_n\} \in S^+$ as claimed in [8].

The Euler numbers $\{E_n\}$ is defined by $\frac{2e^t}{e^{2t}+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}$ ($|t| < \frac{\pi}{2}$), which is equivalent to (see [3]) $E_0 = 1$, $E_{2n-1} = 0$ and $\sum_{r=0}^n \binom{2n}{2r} E_{2r} = 0$ ($n \geq 1$). It is clear that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{E_n - 1}{2^n} \cdot \frac{t^n}{n!} &= \sum_{n=0}^{\infty} E_n \frac{(t/2)^n}{n!} - \sum_{n=0}^{\infty} \frac{(t/2)^n}{n!} \\ &= \frac{2e^{\frac{t}{2}}}{e^t + 1} - e^{\frac{t}{2}} = e^{\frac{t}{2}} \cdot \frac{1 - e^t}{1 + e^t} \quad (|t| < \pi). \end{aligned}$$

As $\frac{1-e^{-t}}{1+e^{-t}} = \frac{e^t-1}{e^t+1}$, we see that $e^{-\frac{t}{2}} \sum_{n=0}^{\infty} \frac{E_n-1}{2^n} \cdot \frac{t^n}{n!}$ is an odd function. Thus $\{\frac{E_n-1}{2^n}\}$ is an odd sequence.

For two numbers b and c , let $\{U_n(b, c)\}$ and $\{V_n(b, c)\}$ be the Lucas sequences given by

$$U_0(b, c) = 0, \quad U_1(b, c) = 1, \quad U_{n+1}(b, c) = bU_n(b, c) - cU_{n-1}(b, c) \quad (n \geq 1)$$

and

$$V_0(b, c) = 2, \quad V_1(b, c) = b, \quad V_{n+1}(b, c) = bV_n(b, c) - cV_{n-1}(b, c) \quad (n \geq 1).$$

It is well known that (see [12]) for $b^2 - 4c \neq 0$,

$$U_n(b, c) = \frac{1}{\sqrt{b^2 - 4c}} \left\{ \left(\frac{b + \sqrt{b^2 - 4c}}{2} \right)^n - \left(\frac{b - \sqrt{b^2 - 4c}}{2} \right)^n \right\}$$

and

$$V_n(b, c) = \left(\frac{b + \sqrt{b^2 - 4c}}{2} \right)^n + \left(\frac{b - \sqrt{b^2 - 4c}}{2} \right)^n.$$

From this one can easily see that for $b(b^2 - 4c) \neq 0$, $\{U_n(b, c)/b^n\}$ is an odd sequence and $\{V_n(b, c)/b^n\}$ is an even sequence.

Let $\{A_n\}$ be an even sequence or an odd sequence. In Section 2 we deduce new recurrence formulas for $\{A_n\}$ and give a criterion for polynomials $P_m(x)$ with the property $P_m(1-x) = (-1)^m P_m(x)$, in Section 3 we establish a transformation formula for $\sum_{k=0}^n \binom{n}{k} A_k$, in Section 4 we give a general congruence involving A_n modulo p^2 , where $p > 3$ is a prime. As applications we establish new recurrence formulas and congruences for Bernoulli and Euler numbers. Here are some typical results:

★ If $\{A_n\}$ is an even sequence and n is odd, then

$$\sum_{k=0}^n \binom{\frac{n}{2}}{k} (-1)^k A_{n-k} = 0 \quad \text{and} \quad \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} (-1)^k A_k = 0.$$

★ If $\{A_n\}$ is an odd sequence, then $\sum_{k=0}^n \binom{n}{k} (-1)^k A_{2n-k} = 0$ for $n = 0, 1, 2, \dots$

★ If $\{A_n\}$ is an even sequence and λ is a real number, then

$$\sum_{k=0}^{2n+1} \binom{2n-\lambda}{2n+1-k} \binom{\lambda}{k} 2^k A_k = 0 \quad (n = 0, 1, 2, \dots).$$

★ Let m be a positive integer and $P_m(x) = \sum_{k=0}^m a_k x^{m-k}$. Then

$$P_m(1-x) = (-1)^m P_m(x) \iff \sum_{k=0}^n \binom{n}{k} \frac{a_k}{\binom{m}{k}} = (-1)^n \frac{a_n}{\binom{m}{n}} \quad (n = 0, 1, \dots, m).$$

★ Let p be an odd prime, and let $\{A_k\}$ be an odd sequence of rational p -integers. Then $\sum_{k=1}^{p-1} \frac{A_k}{p+k} \equiv 0 \pmod{p^2}$.

★ Suppose that $\{a_n\} \in S^+$ with $a_0 \neq 0$ and $A_n = \frac{1}{(n+1)(n+2)} \sum_{k=0}^n a_k \quad (n \geq 0)$. Then $\{A_n\} \in S^+$.

★ Let $[x]$ be the greatest integer not exceeding x . For $n = 3, 4, 5, \dots$ we have

$$\sum_{r=1}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n}{2r-1} (2n-2r+1) B_{2n-2r} = 0 \quad \text{and} \quad \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} 2^{2k} E_{2n-2k} = (-1)^n.$$

In addition to the above notation throughout this paper we use the following notation: \mathbb{N} —the set of positive integers, \mathbb{R} —the set of real numbers, \mathbb{Z}_p —the set of those rational numbers whose denominator is coprime to p , $(\frac{a}{p})$ —the Legendre symbol.

2. Recurrence formulas for even and odd sequences

For $x, y \in \mathbb{R}$ and $n \in \{0, 1, 2, \dots\}$ it is well known that

$$\sum_{k=0}^n \binom{x}{k} \binom{y}{n-k} = \binom{x+y}{n}.$$

This is called Vandermonde's identity. Let $a_n = \sum_{k=0}^n \binom{n-m}{k} (-1)^{n-k} b_{n-k} \quad (n = 0, 1, 2, \dots)$. Using Vandermonde's identity we see that

$$\begin{aligned} & \sum_{k=0}^n \binom{n-m}{k} (-1)^{n-k} a_{n-k} \\ &= \sum_{k=0}^n \binom{n-m}{k} (-1)^{n-k} \sum_{j=0}^{n-k} \binom{n-k-m}{j} (-1)^{n-k-j} b_{n-k-j} \\ &= \sum_{s=0}^n \binom{n-m}{n-s} (-1)^s \sum_{j=0}^s \binom{s-m}{j} (-1)^{s-j} b_{s-j} \\ &= \sum_{s=0}^n \binom{n-m}{n-s} \sum_{r=0}^s \binom{m-r-1}{s-r} b_r \\ &= \sum_{r=0}^n \sum_{s=r}^n \binom{n-m}{n-s} \binom{m-r-1}{s-r} b_r \\ &= \sum_{r=0}^n \binom{n-r-1}{n-r} b_r = b_n \quad (n = 0, 1, 2, \dots). \end{aligned}$$

Lemma 2.1. Let $m, p \in \mathbb{R}$ and $\sum_{k=0}^n \binom{n-m}{k} (-1)^{n-k} a_{n-k} = \pm a_n$ for $n = 0, 1, 2, \dots$. Then

$$\sum_{k=0}^n \binom{n-p-m}{k} (-1)^{n-k} a_{n-k} = \pm \sum_{k=0}^n \binom{p}{k} (-1)^k a_{n-k} \quad \text{for } n = 0, 1, 2, \dots$$

Proof. Using Vandermonde's identity we see that

$$\begin{aligned} & \sum_{k=0}^n \binom{n-p-m}{k} (-1)^{n-k} a_{n-k} \\ &= \sum_{k=0}^n \binom{n-p-m}{n-k} (-1)^k a_k \\ &= \pm \sum_{k=0}^n \binom{n-p-m}{n-k} (-1)^k \sum_{r=0}^k \binom{k-m}{k-r} (-1)^r a_r \\ &= \pm \sum_{r=0}^n \left\{ \sum_{k=r}^n \binom{n-p-m}{n-k} (-1)^{k-r} \binom{k-m}{k-r} \right\} a_r \\ &= \pm \sum_{r=0}^n \left\{ \sum_{k=r}^n \binom{n-p-m}{n-k} \binom{m-1-r}{k-r} \right\} a_r \\ &= \pm \sum_{r=0}^n \left\{ \sum_{s=0}^{n-r} \binom{n-p-m}{n-r-s} \binom{m-1-r}{s} \right\} a_r \\ &= \pm \sum_{r=0}^n \binom{n-p-r-1}{n-r} a_r = \pm \sum_{r=0}^n \binom{p}{n-r} (-1)^{n-r} a_r \\ &= \pm \sum_{k=0}^n \binom{p}{k} (-1)^k a_{n-k}. \end{aligned}$$

So the lemma is proved.

Theorem 2.1. Let $\{A_n\}$ be an even sequence. For $n = 1, 3, 5, \dots$ we have

$$\sum_{k=0}^n \binom{\frac{n}{2}}{k} (-1)^k A_{n-k} = 0.$$

Proof. Putting $m = 0$, $p = n/2$ and $a_n = A_n$ in Lemma 2.1 we deduce the result.

Corollary 2.1. For $n = 1, 3, 5, \dots$ we have

$$\sum_{k=0}^n \binom{\frac{n}{2}}{k} B_{n-k} = 0 \quad \text{and} \quad \sum_{k=0}^n \binom{\frac{n}{2}}{k} \frac{(2^{n-k+1} - 1) B_{n-k+1}}{n - k + 1} = 0.$$

Proof. From [8] we know that $\{(-1)^n B_n\} \in S^+$ and $\{(-1)^{n+1} (2^{n+1} - 1) B_{n+1} / (n+1)\} \in S^+$. Thus the result follows from Theorem 2.1.

Lemma 2.2. If $\{a_n\} \in S^+$, then $\{na_{n-1}\}, \{\frac{a_{n+1}}{n+1}\} \in S^-$. If $\{a_n\} \in S^-$, then $\{na_{n-1}\}, \{\frac{a_{n+1}}{n+1}\} \in S^+$.

Proof. Suppose that $\sum_{k=0}^n \binom{n}{k} (-1)^k a_k = \pm a_n$ for $n = 0, 1, 2, \dots$. Since

$$\begin{aligned} \sum_{k=0}^n \binom{n}{k} (-1)^k k a_{k-1} &= \sum_{k=1}^n n \binom{n-1}{k-1} (-1)^k a_{k-1} \\ &= -n \sum_{r=0}^{n-1} \binom{n-1}{r} (-1)^r a_r = \mp n a_{n-1} \end{aligned}$$

and

$$\begin{aligned} \sum_{k=0}^n \binom{n}{k} (-1)^k \frac{a_{k+1}}{k+1} &= \frac{1}{n+1} \sum_{k=0}^n \binom{n+1}{k+1} (-1)^k a_{k+1} \\ &= -\frac{1}{n+1} \sum_{r=0}^{n+1} \binom{n+1}{r} (-1)^r a_r = \mp \frac{a_{n+1}}{n+1}, \end{aligned}$$

we see that the result is true.

Theorem 2.2. *If $\{A_n\}$ is an odd sequence, then*

$$\sum_{k=0}^n \binom{n}{k} (-1)^k A_{2n-k} = 0 \quad \text{for } n = 0, 1, 2, \dots$$

If $\{A_n\}$ is an even sequence, for $n = 0, 1, 2, \dots$ we have

$$\sum_{k=0}^n \binom{n}{k} (-1)^k (2n-k) A_{2n-k-1} = 0 \quad \text{and} \quad \sum_{k=0}^n \binom{n}{k} (-1)^k \frac{A_{2n-k+1}}{2n-k+1} = 0.$$

Proof. We first assume that $\{A_n\}$ is an odd sequence. Putting $m = 0$, $p = n/2$ and $a_n = A_n$ in Lemma 2.1 we see that

$$\sum_{k=0}^n \binom{\frac{n}{2}}{k} (-1)^{n-k} A_{n-k} = - \sum_{k=0}^n \binom{\frac{n}{2}}{k} (-1)^k A_{n-k}.$$

Thus, for even n we have

$$\sum_{k=0}^{n/2} \binom{\frac{n}{2}}{k} (-1)^k A_{n-k} = \sum_{k=0}^n \binom{\frac{n}{2}}{k} (-1)^k A_{n-k} = 0.$$

Replacing n with $2n$ we get $\sum_{k=0}^{2n} \binom{2n}{k} (-1)^k A_{2n-k} = 0$.

Now we assume that $\{A_n\} \in S^+$. By Lemma 2.2, $\{nA_{n-1}\}, \{\frac{A_{n+1}}{n+1}\} \in S^-$. Thus applying the above we deduce the remaining result.

Corollary 2.2. *For $n = 0, 1, 2, \dots$ we have*

$$\sum_{k=0}^{[n/2]} \binom{n}{2k} 2^{2k} E_{2n-2k} = (-1)^n.$$

Proof. As $\{\frac{E_n-1}{2^n}\} \in S^-$, taking $A_n = \frac{E_n-1}{2^n}$ in Theorem 2.2 we obtain

$$\sum_{k=0}^n \binom{n}{k} (-1)^k \frac{E_{2n-k}-1}{2^{2n-k}} = 0.$$

That is,

$$\sum_{k=0}^n \binom{n}{k} (-2)^k E_{2n-k} = \sum_{k=0}^n \binom{n}{k} (-2)^k = (-1)^n.$$

To see the result, we note that $E_{2m-1} = 0$ for $m \geq 1$.

Corollary 2.3. *For $n = 3, 4, 5, \dots$ we have*

$$\sum_{r=1}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n}{2r-1} (2n-2r+1) B_{2n-2r} = 0.$$

Proof. As $\{(-1)^n B_n\} \in S^+$, taking $A_n = (-1)^n B_n$ in Theorem 2.2 we see that $\sum_{k=0}^n \binom{n}{k} (2n-k) B_{2n-k-1} = 0$. To see the result, we note that $B_{2m+1} = 0$ for $m \geq 1$.

Corollary 2.4. *For $n = 2, 3, 4, \dots$ we have*

$$\sum_{r=0}^{\lfloor n/2 \rfloor} \binom{n}{2r} (2^{2n-2r} - 1) B_{2n-2r} = 0.$$

Proof. The Bernoulli polynomials $\{B_n(x)\}$ are given by $B_n(x) = \sum_{k=0}^n \binom{n}{k} B_k x^{n-k}$. It is well known ([3]) that $B_n(1-x) = (-1)^n B_n(x)$. From [3, p.248] we also have $B_{2k}(\frac{1}{2}) = (2^{1-2k} - 1) B_{2k}$. Thus $B_{2k+1}(\frac{1}{2}) = 0$ and so $B_n(\frac{1}{2}) = (2^{1-n} - 1) B_n$. Hence,

$$\begin{aligned} \sum_{k=0}^n \binom{n}{k} (2^k - 1) B_k &= 2^n \sum_{k=0}^n \binom{n}{k} B_k \left(\frac{1}{2}\right)^{n-k} - \sum_{k=0}^n \binom{n}{k} B_k \\ &= 2^n B_n \left(\frac{1}{2}\right) - (-1)^n B_n = (2 - 2^n) B_n - (-1)^n B_n \\ &= (-1)^n (1 - 2^n) B_n. \end{aligned}$$

That is, $\{(-1)^n (2^n - 1) B_n\}$ is an odd sequence. Now applying Theorem 2.2 we obtain $\sum_{k=0}^n \binom{n}{k} (2^{2n-k} - 1) B_{2n-k} = 0$. To see the result, we note that $B_{2m+1} = 0$ for $m \geq 1$.

Lemma 2.3. *Let $m, n \in \mathbb{N}$ with $m \leq n$. Then*

$$\sum_{k=m}^n \binom{n}{k} \binom{n+k}{k} (-1)^{n-k} \binom{k}{m} = \binom{n}{m} \binom{m+n}{m}.$$

Proof. Using Vandermonde's identity we see that

$$\sum_{k=0}^n \binom{n-m}{n-k} \binom{-n-1}{k} = \binom{-m-1}{n}.$$

Since $\binom{x}{k} = (-1)^k \binom{x+k-1}{k}$, we have

$$\sum_{k=0}^n \binom{n-m}{n-k} (-1)^k \binom{n+k}{k} = (-1)^n \binom{m+n}{n}.$$

That is,

$$\sum_{k=m}^n \binom{n-m}{k-m} (-1)^{n-k} \binom{n+k}{k} = \binom{m+n}{n}.$$

Hence

$$\sum_{k=m}^n \binom{n}{m} \binom{n-m}{k-m} (-1)^{n-k} \binom{n+k}{k} = \binom{n}{m} \binom{m+n}{m}.$$

As $\binom{n}{k} \binom{k}{m} = \binom{n}{m} \binom{n-m}{k-m}$, we obtain the result.

Lemma 2.4. *Let $\{a_k\}$ be a given sequence. For $n \in \mathbb{N}$ we have*

$$\sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} \left(a_k - (-1)^{n-k} \sum_{s=0}^k \binom{k}{s} a_s \right) = 0.$$

Proof. Using Lemma 2.3 we see that

$$\begin{aligned} & \sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} \left(a_k - (-1)^{n-k} \sum_{s=0}^k \binom{k}{s} a_s \right) \\ &= \sum_{m=0}^n a_m \left(\binom{n}{m} \binom{n+m}{m} - \sum_{k=m}^n \binom{n}{k} \binom{n+k}{k} (-1)^{n-k} \binom{k}{m} \right) \\ &= \sum_{m=0}^n a_m \cdot 0 = 0. \end{aligned}$$

Theorem 2.3. *If $\{A_n\}$ is an even sequence and n is odd, or if $\{A_n\}$ is an odd sequence and n is even, then*

$$\sum_{k=0}^n \binom{n}{k} \binom{n+k}{k} (-1)^k A_k = 0.$$

Proof. Putting $a_k = (-1)^k A_k$ in Lemma 2.4 we obtain the result.

Theorem 2.4. *Suppose that m is a nonnegative integer. Then*

$$\sum_{k=0}^n \binom{n-m-1}{k} (-1)^{n-k} a_{n-k} = \pm a_n \quad (n = 0, 1, 2, \dots)$$

if and only if

$$\sum_{k=0}^n \binom{n}{k} \frac{a_k}{\binom{m}{k}} = \pm (-1)^n \frac{a_n}{\binom{m}{n}} \quad (n = 0, 1, \dots, m)$$

and

$$\sum_{k=0}^n \binom{n}{k} (-1)^k a_{k+m+1} = \pm (-1)^{m+1} a_{n+m+1} \quad (n = 0, 1, 2, \dots).$$

Proof. For $n = 0, 1, \dots, m$ we have $\binom{m}{n} \neq 0$. Set $A_n = (-1)^n \frac{a_n}{\binom{m}{n}}$. As $\binom{n-m-1}{k} \binom{m}{n-k} = (-1)^k \binom{n}{k} \binom{m}{n}$, we see that

$$\sum_{k=0}^n \binom{n-m-1}{k} (-1)^{n-k} a_{n-k}$$

$$\begin{aligned}
&= \sum_{k=0}^n \binom{n-m-1}{k} \binom{m}{n-k} A_{n-k} = \binom{m}{n} \sum_{k=0}^n \binom{n}{k} (-1)^k A_{n-k} \\
&= (-1)^n \binom{m}{n} \sum_{k=0}^n \binom{n}{k} (-1)^k A_k.
\end{aligned}$$

Thus,

$$(2.1) \quad \sum_{k=0}^n \binom{n-m-1}{k} (-1)^{n-k} a_{n-k} = \pm a_n \iff \sum_{k=0}^n \binom{n}{k} (-1)^k A_k = \pm A_n.$$

This together with the fact that

$$\begin{aligned}
&\sum_{k=0}^{n+m+1} \binom{n+m+1-m-1}{k} (-1)^{n+m+1-k} a_{n+m+1-k} \\
&= \sum_{k=0}^n \binom{n}{k} (-1)^{n-k+m+1} a_{n-k+m+1} \\
&= \sum_{r=0}^n \binom{n}{r} (-1)^{r+m+1} a_{r+m+1} \quad (n = 0, 1, 2, \dots)
\end{aligned}$$

yields the result.

For any sequence $\{a_n\}$ the formal power series $\sum_{n=0}^{\infty} a_n x^n$ is called the generating function of $\{a_n\}$.

Lemma 2.5. *Let $\{a_n\}$ be a given sequence, $a(x) = \sum_{n=0}^{\infty} a_n x^n$ and $m \in \mathbb{R}$. Then*

$$\begin{aligned}
(1-x)^m a\left(\frac{x}{x-1}\right) &= \pm a(x) \\
\iff \sum_{k=0}^n \binom{n-m-1}{k} (-1)^{n-k} a_{n-k} &= \pm a_n \quad (n = 0, 1, 2, \dots).
\end{aligned}$$

Proof. Clearly,

$$\begin{aligned}
(1-x)^m a\left(\frac{x}{x-1}\right) &= \sum_{r=0}^{\infty} (-1)^r a_r x^r (1-x)^{m-r} \\
&= \sum_{r=0}^{\infty} (-1)^r a_r x^r \sum_{k=0}^{\infty} \binom{m-r}{k} (-x)^k \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n (-1)^{n-k} a_{n-k} \binom{m-(n-k)}{k} (-1)^k \right) x^n \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \binom{n-m-1}{k} (-1)^{n-k} a_{n-k} \right) x^n.
\end{aligned}$$

Thus the result follows.

Theorem 2.5. *Let $m \in \mathbb{N}$, $P_m(x) = \sum_{k=0}^m a_k x^{m-k}$ and $P_m^*(x) = \sum_{k=0}^m a_k x^k$. Then the following statements are equivalent:*

$$(a) \quad (1-x)^m P_m^*\left(\frac{x}{x-1}\right) = \pm P_m^*(x).$$

- (b) $P_m(1-x) = \pm(-1)^m P_m(x)$.
(c) For $n = 0, 1, \dots, m$ we have $\sum_{k=0}^n \binom{n-m-1}{k} (-1)^{n-k} a_{n-k} = \pm a_n$.
(d) Set $a_n = 0$ for $n > m$. Then $\sum_{k=0}^n \binom{n-m-1}{k} (-1)^{n-k} a_{n-k} = \pm a_n$ ($n = 0, 1, 2, \dots$).
(e) For $n = 0, 1, \dots, m$ we have

$$\sum_{k=0}^n \binom{n}{k} \frac{a_k}{\binom{m}{k}} = \pm(-1)^n \frac{a_n}{\binom{m}{n}}.$$

Proof. Since $P_m^*(x) = x^m P_m(\frac{1}{x})$ we see that

$$\begin{aligned} (1-x)^m P_m^*\left(\frac{x}{x-1}\right) &= \pm P_m^*(x) \iff (-x)^m P_m\left(1 - \frac{1}{x}\right) = \pm x^m P_m\left(\frac{1}{x}\right) \\ &\iff \frac{1}{x^m} ((-1)^m P_m(1-x) \mp P_m(x)) = 0 \\ &\iff P_m(1-x) = \pm(-1)^m P_m(x). \end{aligned}$$

So (a) and (b) are equivalent. By Lemma 2.5, (a) is equivalent to (d). Assume $a_{n+m+1} = 0$ for $n \geq 0$. Then

$$\begin{aligned} a_{n+m+1} &= 0 = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k+m+1} a_{m+1+n-k} \\ &= \sum_{k=0}^{m+n+1} \binom{m+n+1-k}{k} (-1)^{m+n+1-k} a_{m+n+1-k}. \end{aligned}$$

So (c) is equivalent to (d). To complete the proof, we note that (d) is equivalent to (e) by Theorem 2.4.

Corollary 2.5. *Let $m \in \mathbb{R}$. Then*

$$\sum_{k=0}^n \binom{n-m-1}{k} (-1)^{n-k} a_{n-k} = \pm a_n \quad (n = 0, 1, 2, \dots)$$

if and only if

$$(2.2) \quad \sum_{k=0}^p \binom{p}{k} (-1)^k \frac{\binom{n-m-1}{n-k}}{\binom{n}{k}} a_k = \pm \frac{\binom{n-m-1}{n-p}}{\binom{n}{p}} a_p$$

for every nonnegative integer n and every $p \in \{0, 1, \dots, n\}$.

Proof. If (2.2) holds, taking $p = n$ in (2.2) we see that

$$(2.3) \quad \sum_{k=0}^n \binom{n-m-1}{k} (-1)^{n-k} a_{n-k} = \pm a_n \quad (n = 0, 1, 2, \dots).$$

Set $P_n(x) = \sum_{k=0}^n \binom{n-m-1}{n-k} (-1)^k a_k x^{n-k}$. Then

$$\begin{aligned} P_n(1-x) &= \sum_{k=0}^n \binom{n-m-1}{k} (-1)^{n-k} a_{n-k} \sum_{r=0}^k \binom{k}{r} (-x)^r \end{aligned}$$

$$\begin{aligned}
&= \sum_{r=0}^n \binom{n-m-1}{r} (-1)^r \sum_{k=r}^n \binom{n-m-1-r}{k-r} (-1)^{n-k} a_{n-k} x^r \\
&= (-1)^n \sum_{r=0}^n \binom{n-m-1}{r} (-1)^{n-r} \left(\sum_{s=0}^{n-r} \binom{n-r-m-1}{s} (-1)^{n-r-s} a_{n-r-s} \right) x^r.
\end{aligned}$$

Thus,

$$\begin{aligned}
(2.4) \quad &P_n(1-x) = \pm(-1)^n P_n(x) \\
&\iff \sum_{s=0}^k \binom{k-m-1}{s} (-1)^{k-s} a_{k-s} = \pm a_k \text{ for } k = 0, 1, \dots, n.
\end{aligned}$$

If (2.3) holds, from (2.4) we see that $P_n(1-x) = \pm(-1)^n P_n(x)$ and so (2.2) holds by Theorem 2.5. This completes the proof.

Theorem 2.6. Suppose that $m, p \in \mathbb{R}$, $\sum_{k=0}^n \binom{n-m}{k} (-1)^{n-k} a_{n-k} = (-1)^\alpha a_n$ and $\sum_{k=0}^n \binom{n-p}{k} (-1)^{n-k} b_{n-k} = (-1)^\beta b_n$ ($n = 0, 1, 2, \dots$). If n is a nonnegative integer such that $\alpha + \beta + n$ is odd, then

$$\sum_{k=0}^n \frac{\binom{n-m}{k} \binom{n-p}{n-k}}{\binom{n}{k}} (-1)^k a_{n-k} b_k = 0.$$

Proof. Set $T_n = \sum_{k=0}^n \frac{\binom{n-m}{k} \binom{n-p}{n-k}}{\binom{n}{k}} (-1)^k a_{n-k} b_k$. From Corollary 2.5 we know that

$$\sum_{r=0}^k \binom{k}{r} (-1)^r \frac{\binom{n-p}{n-r}}{\binom{n}{r}} b_r = (-1)^\beta \frac{\binom{n-m}{n-k}}{\binom{n}{k}} b_k$$

for every nonnegative integer n and $k \in \{0, 1, \dots, n\}$. Thus,

$$\begin{aligned}
T_n &= \sum_{k=0}^n \binom{n-m}{k} (-1)^k a_{n-k} \cdot (-1)^\beta \sum_{r=0}^k \binom{k}{r} (-1)^r \frac{\binom{n-p}{n-r}}{\binom{n}{r}} b_r \\
&= (-1)^\beta \sum_{r=0}^n \frac{\binom{n-p}{n-r}}{\binom{n}{r}} b_r \sum_{k=r}^n \binom{n-m}{k} \binom{k}{r} (-1)^{k-r} a_{n-k} \\
&= (-1)^\beta \sum_{r=0}^n \frac{\binom{n-p}{n-r}}{\binom{n}{r}} b_r \binom{n-m}{r} \sum_{k=r}^n \binom{n-m-r}{k-r} (-1)^{k-r} a_{n-k} \\
&= (-1)^\beta \sum_{r=0}^n \frac{\binom{n-m}{r} \binom{n-p}{n-r}}{\binom{n}{r}} b_r \sum_{s=0}^{n-r} \binom{n-r-m}{s} (-1)^s a_{n-r-s} \\
&= (-1)^{\alpha+\beta+n} \sum_{r=0}^n \frac{\binom{n-m}{r} \binom{n-p}{n-r}}{\binom{n}{r}} (-1)^r a_{n-r} b_r \\
&= (-1)^{\alpha+\beta+n} T_n \quad (n = 0, 1, 2, \dots).
\end{aligned}$$

Hence $T_n = 0$ when $\alpha + \beta + n$ is odd. This completes the proof.

Theorem 2.7. Let λ be a real number.

(i) If $\{a_n\}, \{A_n\} \in S^+$ or $\{a_n\}, \{A_n\} \in S^-$, then

$$\sum_{k=0}^{2n+1} \binom{2n-\lambda}{2n+1-k} \binom{\lambda}{k} a_{2n+1-k} A_k = 0 \quad (n = 0, 1, 2, \dots).$$

(ii) If $\{a_n\} \in S^+$ and $\{A_n\} \in S^-$, then

$$\sum_{k=0}^{2n} \binom{2n-1-\lambda}{2n-k} \binom{\lambda}{k} a_{2n-k} A_k = 0 \quad (n = 0, 1, 2, \dots).$$

Proof. Suppose $\sum_{k=0}^n \binom{n}{k} (-1)^k a_k = (-1)^\alpha a_n$ and $\sum_{k=0}^n \binom{n}{k} (-1)^k A_k = (-1)^\beta A_n$ for $n = 0, 1, 2, \dots$. Set $b_n = (-1)^n \binom{\lambda}{n} A_n$. By (2.1) we have

$$\sum_{k=0}^n \binom{n-1-\lambda}{k} (-1)^{n-k} b_{n-k} = (-1)^\beta b_n.$$

Now taking $m = 0$ and $p = \lambda + 1$ in Theorem 2.6 we see that if $2 \nmid \alpha + \beta + n$, then

$$\sum_{k=0}^n \binom{n-1-\lambda}{n-k} a_{n-k} \binom{\lambda}{k} A_k = \sum_{k=0}^n \binom{n-1-\lambda}{n-k} (-1)^k a_{n-k} b_k = 0.$$

This yields the result.

Corollary 2.6. *Let λ be a real number.*

(i) If $\{A_n\} \in S^+$, then

$$\sum_{k=0}^{2n+1} \binom{2n+1-\lambda}{2n+2-k} \binom{\lambda}{k} A_k = 0 \quad (n = 0, 1, 2, \dots).$$

(ii) If $\{A_n\} \in S^-$, then

$$\sum_{k=0}^{2n} \binom{2n-\lambda}{2n+1-k} \binom{\lambda}{k} A_k = 0 \quad (n = 0, 1, 2, \dots).$$

Proof. As $\{\frac{1}{n+1}\} \in S^+$, putting $a_n = \frac{1}{n+1}$ in Theorem 2.7 we deduce the result.

Corollary 2.7. *Let λ be a real number.*

(i) If $\{A_n\} \in S^+$, then

$$\sum_{k=0}^{2n+1} \binom{2n-\lambda}{2n+1-k} \binom{\lambda}{k} 2^k A_k = 0 \quad (n = 0, 1, 2, \dots).$$

(ii) If $\{A_n\} \in S^-$, then

$$\sum_{k=0}^{2n} \binom{2n-1-\lambda}{2n-k} \binom{\lambda}{k} 2^k A_k = 0 \quad (n = 0, 1, 2, \dots).$$

Proof. As $\{\frac{1}{2^n}\} \in S^+$, putting $a_n = \frac{1}{2^n}$ in Theorem 2.7 we deduce the result.

Theorem 2.8. Let $\{A_n\} \in S^+$ with $A_0 = \dots = A_{l-1} = 0$ and $A_l \neq 0$ ($l \geq 1$). Then

$$\left\{ \frac{A_{n+l}}{(n+1)(n+2) \cdots (n+l)} \right\} \in S^+.$$

Proof. Assume $a_n = A_{n+l}$. Let $a(x)$ and $A(x)$ be the generating functions of $\{a_n\}$ and $\{A_n\}$ respectively. Then clearly $A(x) = x^l a(x)$. Since $A_l = \sum_{k=0}^l \binom{l}{k} (-1)^k A_k = (-1)^l A_l$ we see that $2 \mid l$. Thus, applying Lemma 2.5 and (2.1) we see that

$$\begin{aligned} \{A_n\} \in S^+ &\Leftrightarrow A\left(\frac{x}{x-1}\right) = (1-x)A(x) \Leftrightarrow a\left(\frac{x}{x-1}\right) = (1-x)^{l+1}a(x) \\ &\Leftrightarrow \sum_{r=0}^n \binom{n+l}{r} (-1)^{n-r} a_{n-r} = a_n \quad (n = 0, 1, 2, \dots) \\ &\Leftrightarrow \left\{ \frac{(-1)^n a_n}{\binom{-l-1}{n}} \right\} \in S^+. \end{aligned}$$

Note that

$$(-1)^n \frac{a_n}{\binom{-l-1}{n}} = \frac{a_n}{\binom{n+l}{l}} = \frac{A_{n+l}}{(n+1)(n+2) \cdots (n+l)} \cdot l!.$$

We then obtain the result.

Theorem 2.9. Suppose that $\{a_n\} \in S^+$ with $a_0 \neq 0$ and $A_n = \frac{1}{(n+1)(n+2)} \sum_{k=0}^n a_k$ ($n \geq 0$). Then $\{A_n\} \in S^+$.

Proof. Let $b_0 = b_1 = 0$ and $b_n = \sum_{k=0}^{n-2} a_k$ ($n \geq 2$). Let $a(x) = \sum_{n=0}^{\infty} a_n x^n$ and $b(x) = \sum_{n=0}^{\infty} b_n x^n$. Then

$$\begin{aligned} b(x) &= \sum_{n=2}^{\infty} \left(\sum_{k=0}^{n-2} a_k \right) x^n = x^2 \sum_{n=0}^{\infty} \left(\sum_{k=0}^n a_k \right) x^n \\ &= \frac{x^2}{1-x} \sum_{n=0}^{\infty} a_n x^n = \frac{x^2}{1-x} a(x). \end{aligned}$$

Thus, by [8, Theorem 3.1] or Lemma 2.5 (with $m = -1$) we have

$$\begin{aligned} \{a_n\} \in S^+ &\Leftrightarrow (1-x)a(x) = a\left(\frac{x}{x-1}\right) \Leftrightarrow (1-x)b(x) = b\left(\frac{x}{x-1}\right) \\ &\Leftrightarrow \{b_n\} \in S^+. \end{aligned}$$

Since $b_0 = b_1 = 0$ and $b_2 = a_0 \neq 0$, applying Theorem 2.8 we find that $\left\{ \frac{b_{n+2}}{(n+1)(n+2)} \right\} \in S^+$. That is $\{A_n\} \in S^+$.

Theorem 2.10. Let F be a given function. If $\{A_n\}$ is an even sequence, then

$$\sum_{k=0}^n \binom{n}{k} (-1)^k A_k \left(\sum_{s=0}^k \binom{k}{s} (-1)^s (F(s) - F(n-s)) \right) = 0 \quad (n = 0, 1, 2, \dots).$$

If $\{A_n\}$ is an odd sequence, then

$$\sum_{k=0}^n \binom{n}{k} (-1)^k A_k \left(\sum_{s=0}^k \binom{k}{s} (-1)^s (F(s) + F(n-s)) \right) = 0 \quad (n = 0, 1, 2, \dots).$$

Proof. Suppose that $\sum_{k=0}^n \binom{n}{k} (-1)^k A_k = \pm A_n$. From [7, Lemma 2.1] we have

$$\sum_{k=0}^n \binom{n}{k} (-1)^k f(k) A_k = \pm \sum_{k=0}^n \binom{n}{k} \left(\sum_{r=0}^k \binom{k}{r} (-1)^r F(n-k+r) \right) A_k,$$

where $f(k) = \sum_{s=0}^k \binom{k}{s} (-1)^s F(s)$. Thus

$$\sum_{k=0}^n \binom{n}{k} (-1)^k A_k \left(f(k) \mp \sum_{s=0}^k \binom{k}{s} (-1)^s F(n-s) \right) = 0.$$

This yields the result.

Corollary 2.8. *If $\{A_n\} \in S^+$, then*

$$\sum_{k=0}^n \binom{n}{k} (-1)^k A_k (1+x)^k (1 - (-1)^n x^{n-k}) = 0 \quad \text{for } n = 0, 1, 2, \dots$$

If $\{A_n\} \in S^-$, then

$$\sum_{k=0}^n \binom{n}{k} (-1)^k A_k (1+x)^k (1 + (-1)^n x^{n-k}) = 0 \quad \text{for } n = 0, 1, 2, \dots$$

Proof. Taking $F(s) = (-x)^s$ in Theorem 2.10 and then applying the binomial theorem we obtain the result.

From [7, (2.5)] we know that

$$\sum_{k=0}^n \binom{n}{k} (-1)^k f(m+k) = \sum_{k=0}^m \binom{m}{k} (-1)^k F(n+k),$$

where $F(r) = \sum_{s=0}^r \binom{r}{s} (-1)^s f(s)$. Hence we have:

Theorem 2.11. *If $\{A_n\}$ is an even sequence, then for any nonnegative integers m and n we have*

$$\sum_{k=0}^n \binom{n}{k} (-1)^k A_{k+m} = \sum_{k=0}^m \binom{m}{k} (-1)^k A_{k+n}.$$

If $\{A_n\}$ is an odd sequence, then for any nonnegative integers m and n we have

$$\sum_{k=0}^n \binom{n}{k} (-1)^k A_{k+m} = - \sum_{k=0}^m \binom{m}{k} (-1)^k A_{k+n}.$$

From [8] we know that $\{1/\binom{n+2r-1}{r}\} \in S^+$ for $r = 1, 2, \dots$. Thus, by Theorem 2.11 we have

$$(2.5) \quad \sum_{k=0}^n \binom{n}{k} (-1)^k \frac{1}{\binom{k+m+2r-1}{r}} = \sum_{k=0}^m \binom{m}{k} (-1)^k \frac{1}{\binom{k+n+2r-1}{r}}.$$

Since $\{\frac{U_n(b,c)}{b^n}\} \in S^-$ and $\{\frac{V_n(b,c)}{b^n}\} \in S^+$ for $b(b^2 - 4c) \neq 0$, by Theorem 2.11 we have

$$(2.6) \quad \sum_{k=0}^n \binom{n}{k} (-1)^k \frac{U_{k+m}(b,c)}{b^{k+m}} = - \sum_{k=0}^m \binom{m}{k} (-1)^k \frac{U_{k+n}(b,c)}{b^{k+n}},$$

$$(2.7) \quad \sum_{k=0}^n \binom{n}{k} (-1)^k \frac{V_{k+m}(b,c)}{b^{k+m}} = \sum_{k=0}^m \binom{m}{k} (-1)^k \frac{V_{k+n}(b,c)}{b^{k+n}}.$$

3. A transformation formula for $\sum_{k=0}^n \binom{n}{k} A_n$

Lemma 3.1 ([8, Theorems 4.1 and 4.2]). *Let f be a given function and $n \in \mathbb{N}$.*

(i) *If $\{A_n\}$ is an even sequence, then*

$$\sum_{k=0}^n \binom{n}{k} \left(f(k) - (-1)^{n-k} \sum_{s=0}^k \binom{k}{s} f(s) \right) A_{n-k} = 0.$$

(ii) *If $\{A_n\}$ is an odd sequence, then*

$$\sum_{k=0}^n \binom{n}{k} \left(f(k) + (-1)^{n-k} \sum_{s=0}^k \binom{k}{s} f(s) \right) A_{n-k} = 0.$$

We remark that a simple proof of Lemma 3.1 was given by Wang[11].

Theorem 3.1. *Let $n \in \mathbb{N}$. If $\{A_m\}$ is an even sequence and n is odd, or if $\{A_m\}$ is an odd sequence and n is even, then*

$$\sum_{\substack{k=0 \\ 3|k}}^n \binom{n}{k} A_{n-k} = \sum_{\substack{k=0 \\ 3|n-k}}^n \binom{n}{k} A_k = \frac{1}{3} \sum_{k=0}^n \binom{n}{k} A_k.$$

Proof. Set $\omega = (-1 + \sqrt{-3})/2$. If $\{A_m\}$ is an even sequence and n is odd, putting $f(k) = \omega^k$ in Lemma 3.1 we obtain

$$\sum_{k=0}^n \binom{n}{k} (\omega^k - (-1)^{n-k} (1 + \omega)^k) A_{n-k} = 0.$$

As $1 + \omega = -\omega^2$, we have $\sum_{k=0}^n \binom{n}{k} (\omega^k + \omega^{2k}) A_{n-k} = 0$. Therefore,

$$\begin{aligned} & 3 \sum_{\substack{k=0 \\ 3|k}}^n \binom{n}{k} A_{n-k} - \sum_{k=0}^n \binom{n}{k} A_k \\ &= \sum_{k=0}^n \binom{n}{k} (1 + \omega^k + \omega^{2k}) A_{n-k} - \sum_{k=0}^n \binom{n}{k} A_{n-k} \\ &= \sum_{k=0}^n \binom{n}{k} (\omega^k + \omega^{2k}) A_{n-k} = 0. \end{aligned}$$

The remaining part can be proved similarly.

Corollary 3.1 (Ramanujan [1,5]). *For $n = 3, 5, 7, \dots$ we have*

$$\sum_{\substack{k=0 \\ 6|k-3}}^n \binom{n}{k} B_{n-k} = \begin{cases} -\frac{n}{6} & \text{if } n \equiv 1 \pmod{6}, \\ \frac{n}{3} & \text{if } n \equiv 3, 5 \pmod{6}. \end{cases}$$

Proof. As $\{(-1)^n B_n\} \in S^+$, taking $A_n = (-1)^n B_n$ in Theorem 3.1 we obtain

$$\begin{aligned} \sum_{\substack{k=0 \\ 3|k}}^n \binom{n}{k} (-1)^{n-k} B_{n-k} &= \frac{1}{3} \sum_{k=0}^n \binom{n}{k} (-1)^k B_k = \frac{1}{3} \left(\sum_{k=0}^n \binom{n}{k} B_k + n \right) \\ &= \frac{1}{3} (n + B_n) = \frac{n}{3}. \end{aligned}$$

To see the result, we note that

$$\sum_{\substack{k=0 \\ 3|k}}^n \binom{n}{k} (-1)^{n-k} B_{n-k} - \sum_{\substack{k=0 \\ 6|k-3}}^n \binom{n}{k} B_k = \begin{cases} -nB_1 = \frac{n}{2} & \text{if } n \equiv 1 \pmod{6}, \\ 0 & \text{if } n \equiv 3, 5 \pmod{6}. \end{cases}$$

Corollary 3.2 (Ramanujan [5]). *For $n = 6, 8, 10, \dots$ we have*

$$\frac{4}{3}(2^n - 1)B_n + \sum_{k=1}^{\lfloor n/6 \rfloor} \binom{n}{6k} (2^{n-6k} - 1)B_{n-6k} = \begin{cases} -\frac{n}{6} & \text{if } n \equiv 4 \pmod{6}, \\ \frac{n}{3} & \text{if } n \equiv 0, 2 \pmod{6}. \end{cases}$$

Proof. Since $\{(-1)^n(2^n - 1)B_n\}$ is an odd sequence, by Theorem 3.1 we have

$$\begin{aligned} & \sum_{\substack{k=0 \\ 3|k}}^n \binom{n}{k} (-1)^{n-k} (2^{n-k} - 1)B_{n-k} \\ &= \frac{1}{3} \sum_{k=0}^n \binom{n}{k} (-1)^k (2^k - 1)B_k = \frac{1}{3} \left(\sum_{k=0}^n \binom{n}{k} (2^k - 1)B_k + n \right) \\ &= \frac{1}{3} (-(-1)^n(2^n - 1)B_n + n) = \frac{n}{3} - \frac{1}{3}(2^n - 1)B_n. \end{aligned}$$

On the other hand,

$$\sum_{\substack{k=0 \\ 3|k}}^n \binom{n}{k} (-1)^{n-k} (2^{n-k} - 1)B_{n-k} - \sum_{\substack{k=0 \\ 6|k}}^n \binom{n}{k} (2^{n-k} - 1)B_{n-k} = \begin{cases} -nB_1 = \frac{n}{2} & \text{if } 6 \mid n - 4, \\ 0 & \text{if } 6 \nmid n - 4. \end{cases}$$

Thus the result follows.

Corollary 3.3 (Lehmer [2]). *For $n = 6, 8, 10, \dots$ we have*

$$E_n + 3 \sum_{k=1}^{\lfloor n/6 \rfloor} \binom{n}{6k} 2^{6k-2} E_{n-6k} = \frac{1 + (-3)^{n/2}}{2}.$$

Proof. Since $\{(E_n - 1)/2^n\}$ is an odd sequence, by Theorem 3.1 and the fact $E_{2k+1} = 0$ we have

$$\begin{aligned} & \sum_{\substack{k=0 \\ 6|k}}^n \binom{n}{k} \frac{E_{n-k}}{2^{n-k}} - \sum_{\substack{k=0 \\ 3|k}}^n \binom{n}{k} \frac{1}{2^{n-k}} = \sum_{\substack{k=0 \\ 3|k}}^n \binom{n}{k} \frac{E_{n-k} - 1}{2^{n-k}} = \frac{1}{3} \sum_{k=0}^n \binom{n}{k} \frac{E_k - 1}{2^k} \\ &= \frac{1}{3} \left\{ \sum_{k=0}^n \binom{n}{k} (-1)^k \frac{E_k - 1}{2^k} + \left(1 - \frac{1}{2}\right)^n - \left(1 + \frac{1}{2}\right)^n \right\} \\ &= \frac{1}{3} \left\{ -\frac{E_n - 1}{2^n} + \frac{1 - 3^n}{2^n} \right\} = \frac{2 - 3^n - E_n}{3 \cdot 2^n}. \end{aligned}$$

Observe that

$$\begin{aligned} \sum_{\substack{k=0 \\ 3|k}}^n \binom{n}{k} 2^k &= \sum_{k=0}^n \binom{n}{k} 2^k \cdot \frac{1}{3} (1 + \omega^k + \omega^{2k}) \\ &= \frac{1}{3} ((1 + 2)^n + (1 + 2\omega)^n + (1 + 2\omega^2)^n) \\ &= \frac{1}{3} (3^n + (\sqrt{-3})^n + (-\sqrt{-3})^n) = \frac{1}{3} (3^n + 2 \cdot (-3)^{\frac{n}{2}}). \end{aligned}$$

We obtain

$$\frac{4}{3}E_n + \sum_{\substack{k=1 \\ 6|k}}^n \binom{n}{k} 2^k E_{n-k} = \frac{2-3^n}{3} + \frac{3^n + 2 \cdot (-3)^{\frac{n}{2}}}{3} = \frac{2}{3}(1 + (-3)^{\frac{n}{2}}).$$

This yields the result.

Remark 3.1 Compared with known proofs of Corollaries 3.1-3.3 (see [1,2,5]), our proofs are simple and natural.

Now we introduce new sequence $\{S_n\}$ defined by

$$(3.1) \quad S_n + \sum_{k=0}^n \binom{n}{k} S_k = 2 \quad (n = 0, 1, 2, \dots).$$

The first few values of S_n are shown below:

$$S_0 = 1, \quad S_1 = \frac{1}{2}, \quad S_3 = -\frac{1}{4}, \quad S_5 = \frac{1}{2}, \quad S_7 = -\frac{17}{8}, \quad S_9 = \frac{31}{2}, \quad S_{11} = -\frac{691}{4}, \\ S_2 = S_4 = S_6 = S_8 = S_{10} = 0.$$

As

$$(1 + e^{-x}) \left(\sum_{n=0}^{\infty} S_n \frac{x^n}{n!} \right) = \sum_{n=0}^{\infty} \left(S_n + \sum_{k=0}^n \binom{n}{k} S_k \right) \frac{x^n}{n!} = 2,$$

we see that

$$(3.2) \quad \sum_{n=0}^{\infty} S_n \frac{x^n}{n!} = \frac{2e^x}{e^x + 1} \quad (|x| < 2\pi).$$

Since $\sum_{n=1}^{\infty} S_n \frac{x^n}{n!} = \frac{e^x - 1}{e^x + 1} = -\frac{e^{-x} - 1}{e^{-x} + 1}$, we have $S_n = -(-1)^n S_n$ and so $S_{2m} = 0$ for $m \geq 1$. As $e^{-x/2} \sum_{n=0}^{\infty} S_n \frac{x^n}{n!} = \frac{2}{e^{x/2} + e^{-x/2}}$ is an even function, we have

$$(3.3) \quad \{S_n\} \in S^+.$$

Observe that $\sum_{n=0}^{\infty} (\sum_{k=0}^n \binom{n}{k} E_k) \frac{x^n}{n!} = e^x \cdot \frac{2e^x}{e^{2x} + 1}$. We also have

$$(3.4) \quad S_n = \frac{1}{2^n} \sum_{k=0}^n \binom{n}{k} E_k = \frac{1}{2^n} \sum_{\substack{k=0 \\ 2|k}}^n \binom{n}{k} E_k.$$

Corollary 3.4. For $n = 1, 3, 5, \dots$ we have

$$4S_n + 3 \sum_{k=1}^{\lfloor n/6 \rfloor} \binom{n}{6k} S_{n-6k} = \begin{cases} 2 & \text{if } 3 \nmid n, \\ -1 & \text{if } 3 \mid n. \end{cases}$$

Proof. As $\{S_n\} \in S^+$, from Theorem 3.1 we see that for odd n ,

$$3S_n + 3 \sum_{\substack{k=1 \\ 3|k}}^n \binom{n}{k} S_{n-k} = 3 \sum_{\substack{k=0 \\ 3|k}}^n \binom{n}{k} S_{n-k} = \sum_{k=0}^n \binom{n}{k} S_k = 2 - S_n.$$

To see the result, we recall that $S_0 = 1$ and $S_{2m} = 0$ for $m \geq 1$.

4. Congruences involving even and odd sequences

Theorem 4.1. *Let p be an odd prime, and let $\{A_k\}$ be an odd sequence of rational p -integers. Then*

$$\sum_{k=1}^{p-1} \frac{A_k}{p+k} \equiv 0 \pmod{p^2}.$$

Proof. Taking $n = p - 1$ in Theorem 2.3 we get

$$\sum_{k=0}^{p-1} \binom{p-1}{k} \binom{p-1+k}{k} (-1)^k A_k = 0.$$

For $k = 1, 2, \dots, p-1$ we see that

$$\begin{aligned} \binom{p-1}{k} \binom{p-1+k}{k} &= \frac{(p-1)(p-2) \cdots (p-k)}{k!} \cdot \frac{p(p+1) \cdots (p+k-1)}{k!} \\ &= \frac{p}{p+k} \cdot \frac{(p^2-1^2)(p^2-2^2) \cdots (p^2-k^2)}{k!^2} \\ &\equiv (-1)^k \frac{p}{p+k} \pmod{p^3}. \end{aligned}$$

Since $A_0 = -A_0$ we have $A_0 = 0$. Now, from all the above we deduce the result.

Remark 4.1 For given odd prime p and odd sequence $\{A_n\}$ of rational p -integers, the congruence $\sum_{k=1}^{p-1} \frac{A_k}{k} \equiv 0 \pmod{p}$ was given by Tauraso[10] earlier.

Corollary 4.1. *Let p be an odd prime, and let $\{A_k\}$ be an odd sequence of rational p -integers. Then*

$$\sum_{k=1}^{p-1} \frac{A_k}{k} \equiv p \sum_{k=1}^{p-1} \frac{A_k}{k^2} \pmod{p^2}.$$

Proof. For $k = 1, 2, \dots, p-1$ we have $\frac{1}{k+p} = \frac{k-p}{k^2-p^2} \equiv \frac{k-p}{k^2} = \frac{1}{k} - \frac{p}{k^2} \pmod{p^2}$. Thus, the result follows from Theorem 4.1.

Corollary 4.2. *Let p be an odd prime. Then*

$$\sum_{k=1}^{(p-1)/2} \frac{(2^{2k}-1)B_{2k}}{p+2k} \equiv \frac{p-1}{2} \pmod{p^2} \quad \text{and} \quad \sum_{k=1}^{(p-1)/2} \frac{E_{2k}-1}{(p+2k)2^{2k}} \equiv 0 \pmod{p^2}.$$

Proof. Since $\{(-1)^n(2^n-1)B_n\}$ and $\{\frac{E_n-1}{2^n}\}$ are odd sequences, $E_{2m-1} = 0$ and $B_{2m+1} = 0$ for $m \geq 1$, the result follows from Theorem 4.1.

Corollary 4.3. *Let p be an odd prime, $b, c \in \mathbb{Z}_p$ and $b(b^2-4c) \not\equiv 0 \pmod{p}$. Then*

$$\sum_{k=0}^{p-1} \frac{U_k(b, c)}{(p+k)b^k} \equiv 0 \pmod{p^2}.$$

Proof. Since $\{\frac{U_n(b, c)}{b^n}\}$ is an odd sequence, the result follows from Theorem 4.1.

Let $F_n = U_n(1, -1)$ and $L_n = V_n(1, -1)$ be the Fibonacci sequence and Lucas sequence, respectively. From Section 1 we know that $\{F_n\}$ is an odd sequence and $\{L_n\}$ is an even sequence.

Corollary 4.4. *Let $p > 5$ be a prime. Then*

$$\sum_{k=1}^{p-1} \frac{F_k}{k} \equiv -\left(\frac{p}{5}\right) \frac{5p}{4} \left(\frac{F_{p-(\frac{p}{5})}}{p}\right)^2 \pmod{p^2}.$$

Proof. Recently Hao Pan and Zhi-Wei Sun ([4]) proved that

$$\sum_{k=1}^{p-1} \frac{F_k}{k^2} \equiv -\frac{1}{5} \left(\frac{p}{5}\right) \left(\frac{L_p - 1}{p}\right)^2 \pmod{p}.$$

It is known ([6]) that $F_{p-(\frac{p}{5})} \equiv 0 \pmod{p}$ and $L_{p-(\frac{p}{5})} \equiv 2(\frac{p}{5}) \pmod{p^2}$. Also, $5F_n = 2L_{n+1} - L_n = L_n + 2L_{n-1}$. Thus

$$5F_{p-(\frac{p}{5})} = 2L_p - \left(\frac{p}{5}\right) L_{p-(\frac{p}{5})} \equiv 2(L_p - 1) \pmod{p^2}$$

and so

$$\sum_{k=1}^{p-1} \frac{F_k}{k^2} \equiv -\frac{1}{5} \left(\frac{p}{5}\right) \left(\frac{5F_{p-(\frac{p}{5})}}{2p}\right)^2 \pmod{p^2}.$$

Since $\{F_k\}$ is an odd sequence, applying Corollary 4.1 we deduce the result.

Theorem 4.2. *Let p be a prime greater than 3, and let $\{A_k\}$ be an even sequence. Suppose that $A_0, A_1, \dots, A_{p-2}, A_p, pA_{p-1} \in \mathbb{Z}_p$. Then*

$$\sum_{k=1}^{p-2} \frac{A_k}{p-k} \equiv \frac{2A_p - A_0 - pA_{p-1}}{p} \pmod{p^2}.$$

Proof. Taking $n = p$ in Theorem 2.3 we get

$$\sum_{k=0}^p \binom{p}{k} \binom{p+k}{k} (-1)^k A_k = 0.$$

For $k = 1, 2, \dots, p-1$ we see that

$$\begin{aligned} \binom{p}{k} \binom{p+k}{k} &= \frac{p(p-1) \cdots (p-k+1)}{k!} \cdot \frac{(p+1) \cdots (p+k)}{k!} \\ &= \frac{p}{p-k} \cdot \frac{(p^2-1^2)(p^2-2^2) \cdots (p^2-k^2)}{k!^2} \\ &\equiv (-1)^k \frac{p}{p-k} \pmod{p^3}. \end{aligned}$$

Thus,

$$A_0 - \binom{2p}{p} A_p + \binom{p}{p-1} \binom{2p-1}{p-1} A_{p-1} + \sum_{k=1}^{p-2} \frac{p}{p-k} A_k$$

$$\equiv \sum_{k=0}^p \binom{p}{k} \binom{p+k}{k} (-1)^k A_k \equiv 0 \pmod{p^3}.$$

Hence

$$\sum_{k=1}^{p-2} \frac{A_k}{p-k} \equiv \frac{2 \binom{2p-1}{p-1} A_p - p \binom{2p-1}{p-1} A_{p-1} - A_0}{p} \pmod{p^2}.$$

The famous Wolstenholme's congruence ([13]) states that $\binom{2p-1}{p-1} \equiv 1 \pmod{p^3}$. Thus the result follows.

Corollary 4.5. *Let p be a prime greater than 3. Then*

$$\sum_{k=1}^{(p-3)/2} \frac{B_{2k}}{p-2k} \equiv \frac{p+1}{2} - \frac{pB_{p-1}+1}{p} \pmod{p^2}.$$

Proof. It is well known that $B_0, B_1, \dots, B_{p-2}, B_p, pB_{p-1} \in \mathbb{Z}_p$. Taking $A_k = (-1)^k B_k$ in Theorem 4.2 and applying the fact $B_{2k+1} = 0$ for $k \geq 1$ we deduce the result.

Corollary 4.6. *Let p be a prime greater than 3, and let $\{A_k\}$ be an even sequence with $A_0, A_1, \dots, A_{p-2}, A_p, pA_{p-1} \in \mathbb{Z}_p$. Then*

$$\sum_{k=1}^{p-2} \frac{A_k}{k} \equiv -p \sum_{k=1}^{p-2} \frac{A_k}{k^2} + \frac{A_0 + pA_{p-1} - 2A_p}{p} \pmod{p^2}.$$

Proof. For $k = 1, 2, \dots, p-2$ we have $\frac{1}{k-p} = \frac{k+p}{k^2-p^2} \equiv \frac{k+p}{k^2} = \frac{1}{k} + \frac{p}{k^2} \pmod{p^2}$. Thus, by Theorem 4.2 we obtain the result.

Corollary 4.7. *Let $p > 3$ be a prime, $b, c \in \mathbb{Z}_p$ and $b(b^2 - 4c) \not\equiv 0 \pmod{p}$. Then*

$$\sum_{k=1}^{p-1} \frac{V_k(b, c)}{(p-k)b^k} \equiv \frac{2(V_p(b, c) - b^p)}{pb^p} \pmod{p^2}.$$

Proof. Taking $A_k = V_k(b, c)/b^k$ in Theorem 4.2 we deduce the result.

Corollary 4.8. *Let $p > 5$ be a prime. Then*

$$\sum_{k=1}^{p-1} \frac{L_k}{k} \equiv \frac{2(1 - L_p)}{p} \pmod{p^2}.$$

Proof. Recently Hao Pan and Zhi-Wei Sun ([4]) proved the following conjecture of Tauraso: $\sum_{k=1}^{p-1} \frac{L_k}{k^2} \equiv 0 \pmod{p}$. Thus taking $A_k = L_k$ in Corollary 4.6 we see that

$$\begin{aligned} \sum_{k=1}^{p-2} \frac{L_k}{k} &\equiv -p \left(\sum_{k=1}^{p-1} \frac{L_k}{k^2} - \frac{L_{p-1}}{(p-1)^2} \right) + \frac{2 + pL_{p-1} - 2L_p}{p} \\ &\equiv (p+1)L_{p-1} + \frac{2(1 - L_p)}{p} \equiv -\frac{L_{p-1}}{p-1} + \frac{2(1 - L_p)}{p} \pmod{p^2}. \end{aligned}$$

This yields the result.

Theorem 4.3. *Let p be an odd prime and $a_0, a_1, \dots, a_{\frac{p-1}{2}} \in \mathbb{Z}_p$. If $\{a_n\}$ is an even sequence and $p \equiv 3 \pmod{4}$, or if $\{a_n\}$ is an odd sequence and $p \equiv 1 \pmod{4}$, then*

$$\sum_{k=0}^{(p-1)/2} \binom{2k}{k}^2 \frac{a_k}{16^k} \equiv \sum_{k=0}^{(p-1)/2} \binom{2k}{k}^2 \frac{a_{k+2} - a_{k+1}}{16^k} \equiv 0 \pmod{p^2}$$

and

$$\sum_{k=1}^{(p-1)/2} \binom{2k}{k}^2 \frac{ka_{k-1}}{16^k} \equiv \sum_{k=0}^{(p-1)/2} \binom{2k}{k}^2 \frac{a_{k+1}}{16^k(k+1)} \equiv 0 \pmod{p^2}.$$

Proof. Suppose that $\{a_n\} \in S^\pm$. By [8, Corollary 3.1], $\{a_{n+2} - a_{n+1}\} \in S^\pm$. By Lemma 2.2, $\{na_{n-1}\} \in S^\mp$ and $\{\frac{a_{n+1}}{n+1}\} \in S^\mp$. From Theorem 2.3 we see that if $\{A_n\} \in S^+$ and $p \equiv 3 \pmod{4}$, or if $\{A_n\} \in S^-$ and $p \equiv 1 \pmod{4}$, then

$$\sum_{k=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}}{k} \binom{\frac{p-1}{2} + k}{k} (-1)^k A_k = 0.$$

By [9, Lemma 2.2], for $k = 0, 1, \dots, \frac{p-1}{2}$ we have $\binom{\frac{p-1}{2} + k}{2k} \equiv \frac{1}{(-16)^k} \binom{2k}{k} \pmod{p^2}$. Thus,

$$\binom{\frac{p-1}{2}}{k} \binom{\frac{p-1}{2} + k}{k} = \binom{2k}{k} \binom{\frac{p-1}{2} + k}{2k} \equiv \frac{1}{(-16)^k} \binom{2k}{k}^2 \pmod{p^2}$$

and so $\sum_{k=0}^{\frac{p-1}{2}} \frac{1}{16^k} \binom{2k}{k}^2 A_k \equiv 0 \pmod{p^2}$ provided that $A_0, A_1, \dots, A_{\frac{p-1}{2}} \in \mathbb{Z}_p$. Now combining all the above we deduce the result.

Corollary 4.9. *Let p be a prime of the form $4k + 3$. Then*

$$\sum_{k=0}^{(p-3)/4} \binom{4k}{2k}^2 \frac{B_{2k}}{16^{2k}} \equiv -\frac{1}{8} \pmod{p^2}.$$

Proof. Since $\{(-1)^n B_n\}$ is an even sequence and $B_{2m+1} = 0$ for $m \geq 1$, taking $a_n = (-1)^n B_n$ in Theorem 4.3 we deduce the result.

Corollary 4.10. *Let p be a prime of the form $4k + 1$ and $p = a^2 + b^2$ with $a, b \in \mathbb{Z}$ and $a \equiv 1 \pmod{4}$. Then*

$$\sum_{k=0}^{(p-1)/4} \binom{4k}{2k}^2 \frac{E_{2k}}{32^{2k}} \equiv 2a - \frac{p}{2a} \pmod{p^2}.$$

Proof. Since $\{\frac{E_{n-1}}{2^n}\}$ is an odd sequence, taking $a_k = (E_k - 1)/2^k$ in Theorem 4.3 and then applying [9, Theorem 2.2] we deduce that

$$\sum_{k=0}^{(p-1)/2} \binom{2k}{k}^2 \frac{E_k}{32^k} \equiv \sum_{k=0}^{(p-1)/2} \binom{2k}{k}^2 \frac{1}{32^k} \equiv 2a - \frac{p}{2a} \pmod{p^2}.$$

To see the result, we note that $E_k = 0$ for odd k .

Theorem 4.4. *Let p be an odd prime and $A_0, A_1, \dots, A_{\frac{p-1}{2}} \in \mathbb{Z}_p$. If $\{A_n\} \in S^+$ and $p \equiv 3 \pmod{4}$, or if $\{A_n\} \in S^-$ and $p \equiv 1 \pmod{4}$, then*

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{2^k} A_k \equiv 0 \pmod{p}.$$

Proof. Since $\{\frac{1}{2^n}\} \in S^+$, by Lemma 3.1(i) we have

$$\sum_{k=0}^{(p-1)/2} \binom{\frac{p-1}{2}}{k} \left((-1)^k a_k - (-1)^{\frac{p-1}{2}-k} \sum_{s=0}^k \binom{k}{s} (-1)^s a_s \right) \frac{1}{2^{\frac{p-1}{2}-k}} = 0.$$

Note that $\binom{\frac{p-1}{2}}{k} \equiv \binom{-\frac{1}{2}}{k} = \frac{1}{(-4)^k} \binom{2k}{k} \pmod{p}$. Taking $a_k = A_k$ in the above we deduce the result.

Theorem 4.5. *Let p be an odd prime and $A_0, A_1, \dots, A_p \in \mathbb{Z}_p$. If $\{A_n\}$ is an odd sequence, then*

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{4^k} A_{p-1-k} \equiv 0 \pmod{p}.$$

If $\{A_n\}$ is an even sequence, then

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k} (k+1)}{4^k} A_{p-2-k} \equiv \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{4^k \cdot k} A_{p-k} \equiv 0 \pmod{p}.$$

Proof. Note that $\binom{\frac{p-1}{2}}{k} \equiv \binom{-\frac{1}{2}}{k} = \frac{1}{(-4)^k} \binom{2k}{k} \pmod{p}$. Taking $n = \frac{p-1}{2}$ in Theorem 2.2 we deduce the result.

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